Ro Jefferson — GCRFI Holography Bootcamp — 22.11.17

Extrapolate dictionary:

$$\lim_{z \to 0} z^{-\Delta} \Phi(t,\vec{x},z) = \mathcal{O}(t,\vec{x})$$

bulk field　　　　bndy operator

→ maps between bulk fields & bndy ops @ $z \to 0$; i.e.,

given CFT operators $\mathcal{O}_i$, can reconstruct bulk fields in $z \to 0$ limit.

Question: How to reconstruct fields deep in the bulk??

Role of "bulk reconstruction":

(1) Make map between bulk & bndy (Φ & $\mathcal{O}$, respect.) precise (flesh-out holographic dictionary).

(2) Given complete knowledge of CFT, reconstruct bulk physics from bndy data.

Implicit philosophy: field theory more fundamental, (gravitational) bulk emergent.

No alternative non-perturbative description of string theory in bulk, thus AdS/CFT defines (via CFT) quantum gravity in AdS (at large-N).

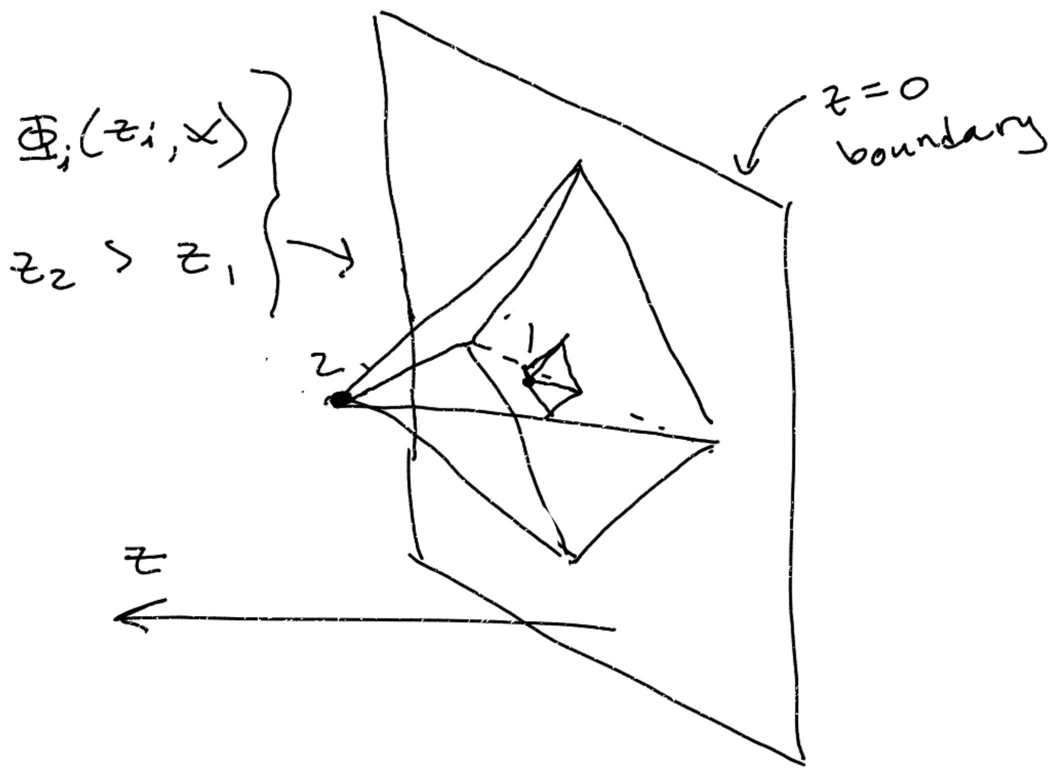
Bulk reconstruction via extrapolate dict. took off with seminal work by Hamilton, Kabat, Lifschytz, + Lowe (HKLL) [hep-th/0606141, 0506118, 0612053].

Intuition:

$$\lim_{z\to 0} z^{-\Delta}\,\Phi(\underset{x=(t,\vec{x})}{x}, z) = \mathcal{O}(x)$$

relates bndy value of local bulk field to local bndy op.

But as we move deeper into bulk, spacelike-separated bndy region grows:



Define Φ as smeared bndy op:

$$\Phi(z,x) = \int_R dy\, K(z,x|y)\, \mathcal{O}(y)$$

$$\cancel{+\left(\frac{1}{N} \text{ corrections}\right)}$$

K = "<u>smearing function</u>"

<u>N.b.</u> $\int_R dy$ over spacelike-separated boundary points y <u>only</u>; cannot be timelike, since then Φ would not commute (<u>causality</u>).

<u>Basic idea</u>:

(1) Expand Φ in creation/annih. modes:

$$\Phi(z,x) = \int dk \; a_k \, \Phi_k(z,x) + h.c. \qquad (*)$$

(2) Take $z \to 0 \Rightarrow \Phi_k \sim z^{-\Delta} \varphi_k$

bndy mode functions

(i.e., extract leading-order behavior as $\Phi(z\to 0, x)$ decays at bndy, where $\Delta = \frac{d}{2} \pm \sqrt{\left(\frac{d}{2}\right)^2 + m^2 \ell^2}$).

$$\Rightarrow \mathcal{O}(x) = \int dk \; a_k \varphi_k + h.c. \qquad (\dagger)$$

(3) (Choose) φ_k orthonormal $\longrightarrow$ invert $(\dagger)$:

$$a_k = \int dx \; \mathcal{O}(x) \, \varphi_k^*(x)$$

then insert into $(*)$ to obtain:

$$\Phi(z,x) = \int dk \int dy \; \varphi_k^*(y)\, \mathcal{O}(y)\, \Phi_k(z,x) + \text{h.c.}$$

(4) Define smearing function

$$K(z,x|y) \equiv \int dk \; \varphi_k^*(y)\, \Phi_k(z,x)$$

$$\Rightarrow \quad \Phi(z,x) = \int dy \; K(z,x|y)\, \mathcal{O}(y)$$

Some comments / caveats:

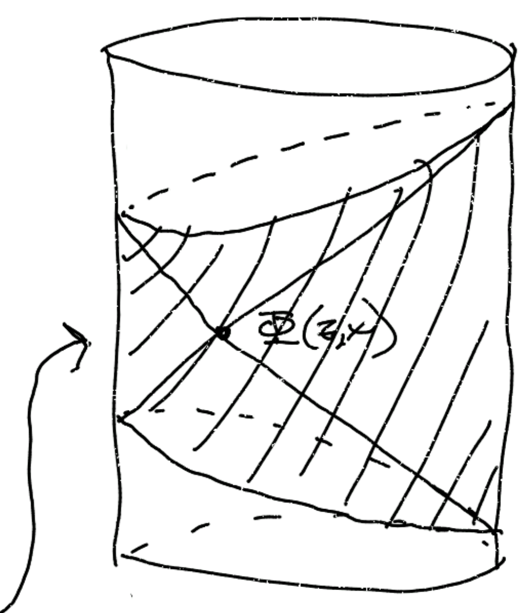
(a) Bulk modes Φ_k must be normalized such that asymptotic bndy modes φ_k orthonormal. This is not always possible! For AdS-Schwarz, angular momentum barrier implies divergence as trace bndy modes back into bulk. (see 1304.6821).

(b) Assumed could swap order

$$\int dk \int dy \longleftrightarrow \int dy \int dk$$

Again, not always true: some cases $\int dk$ diverges, cannot define K in real-space (but can work with momentum-space K; see below).

Domain of K is spacelike w.r.t. Φ:



global AdS_{d+1}:

$$ds^2 = \frac{\ell^2}{\cos^2\rho}\left(-dt^2 + d\rho^2 + \sin^2\rho \, d\Omega^2_{d-1}\right),$$

$$-\infty < t < \infty, \quad 0 \le \rho < \pi/2$$

integrate K over

Let's build K in global AdS_{d+1}

(for free scalar fields):

- Expand soln's to $(\Box - m^2)\Phi = 0$ in normalizable modes:

$$\Phi(t, \overbrace{\rho, \Omega}^{x}) = \sum_{n=0}^{\infty} \sum_{\ell, m} a_{n\ell m}\, e^{-i(2n+\ell+\Delta)t}$$

$$\times \cos^{\Delta}\rho\, \sin^{\ell}\rho\, P_n^{(\Delta - \frac{d}{2},\, \ell + \frac{d}{2} - 1)}(-\cos 2\rho)\, Y_{\ell m}(\Omega) + c.c.$$

Jacobi poly $P_n^{(\alpha, \beta)}$

spherical harmonic

Bndy value $\displaystyle \Phi_0(t, \Omega) = \lim_{\rho \to \pi/2} \frac{\Phi(\rho, t, \Omega)}{\cos^{\Delta}\rho}$

- Simplify: consider $\rho = 0$ (can later use AdS isometries to get arbitrary bulk point)

$$\Rightarrow \quad \Phi(t, 0, \Omega) = \sum_{n=0}^{\infty} a_n\, e^{-i(2n+\Delta)}\, P_n^{(\Delta - \frac{d}{2},\, \frac{d}{2} - 1)}(-1) + c.c.$$

(so now only s-waves).

- Split $\Phi_0 = \Phi_0^{+} + \Phi_0^{-}$

$$\Phi_0^{+} = \sum_{n=0}^{\infty} a_n\, e^{-i(2n+\Delta)t}\, P_n^{(\Delta - \frac{d}{2},\, \frac{d}{2} - 1)}, \quad \Phi_0^{-} = \left(\Phi_0^{+}\right)^{*}$$

with Fourier components

$$a_n = \frac{1}{\pi \, \mathrm{Vol}(S^{d-1}) \, P_n^{(\Delta - \frac{d}{2}, \frac{d}{2} - 1)}}$$

$$\times \int_{-\pi/2}^{\pi/2} d\tau \int d\Omega \sqrt{g_\Omega} \; e^{i(2n+\Delta)t} \; \Phi_0^+(t)$$

- Plug back into $\Phi(\rho=0, t, \Omega) =$

$$\int_{-\pi/2}^{\pi/2} dt' \int d\Omega' \sqrt{g_\Omega} \; K_+(\rho=0, t, \Omega \mid t', \Omega') \; \Phi_0^+(t', \Omega') + \text{c.c.}$$

$$\boxed{K_+ = \frac{1}{\pi \, \mathrm{Vol}(S^{d-1})} \; e^{i\Delta t} \; {}_2F_1\left(1, \frac{d}{2}, \Delta - \frac{d}{2} + 1, -e^{2it}\right)}$$

For general $\Phi(\rho)$: see HKLL hep-th/0606141

In principle, can now reconstruct bulk fields given enough bndy data.

$\hookrightarrow$ by extension, bulk correlators:

$$\langle \Phi(z_1, x_1) \Phi(z_2, x_2) \rangle = \int dy_1 \, dy_2 \; K(z_1, x_1 \mid y_1)$$

$$\times \; K(z_2, x_2 \mid y_2) \langle \mathcal{O}(y_1) \mathcal{O}(y_2) \rangle$$

Question: how much of bndy do we really need? That is, what is minimum amount of bndy info needed to encode given bulk feature? (Want to make holo. dictionary as precise as possible).

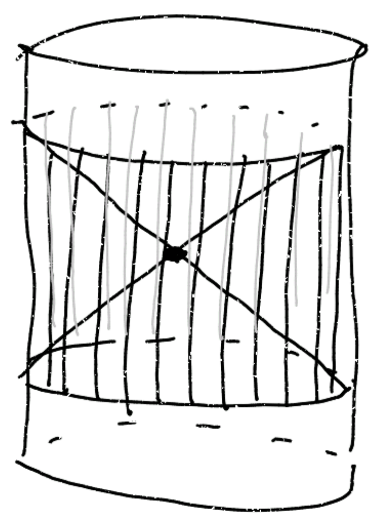
Global:

$$ds^2 = \frac{\ell^2}{\cos^2\rho}\left(-dt^2 + d\rho^2 + \sin^2\rho \, d\Omega_{d-1}^2\right)$$

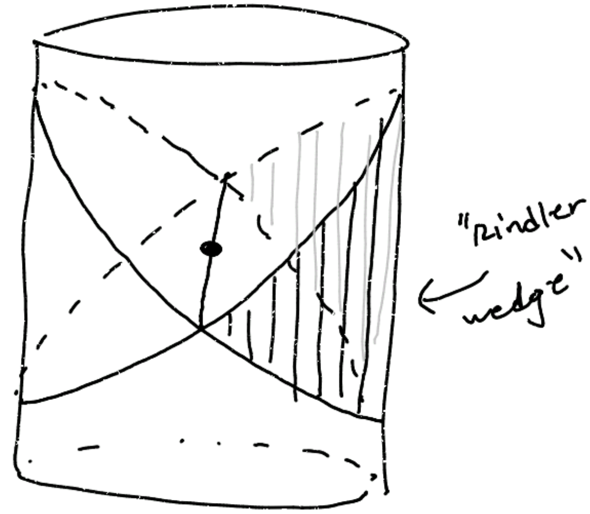
AdS-Rindler

$$ds^2 = -f(r)\,dt^2 + \frac{1}{f(r)}\,dr^2 + r^2 d\Omega^2$$

$$f(r) = \frac{r^2 - r_+^2}{\ell^2}$$

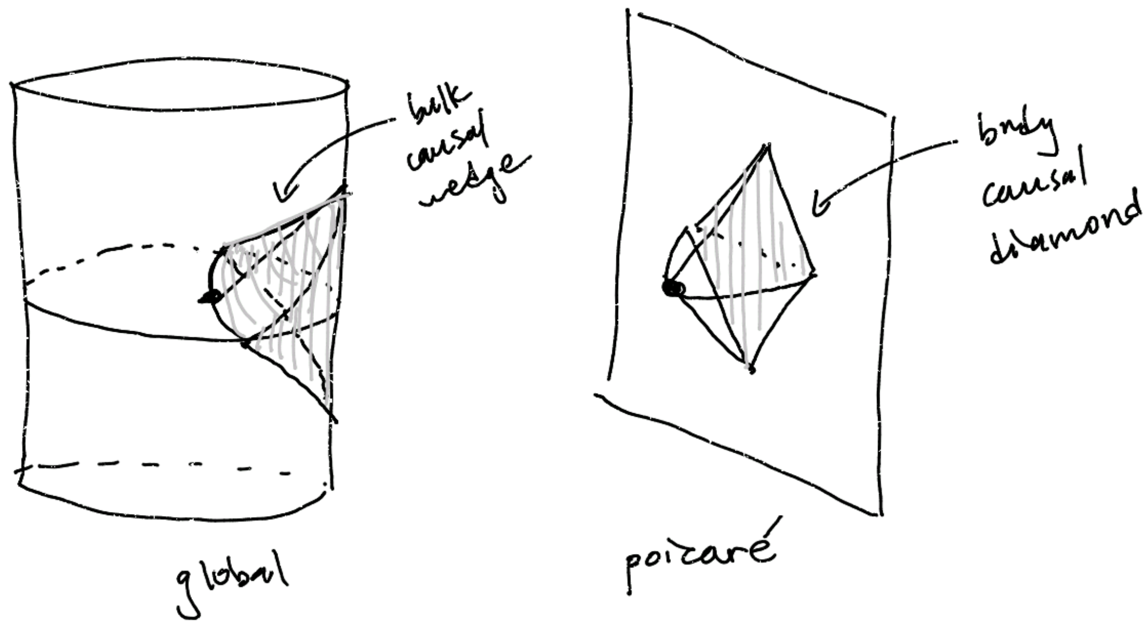


bndy support of local bulk op entirely non local on bndy



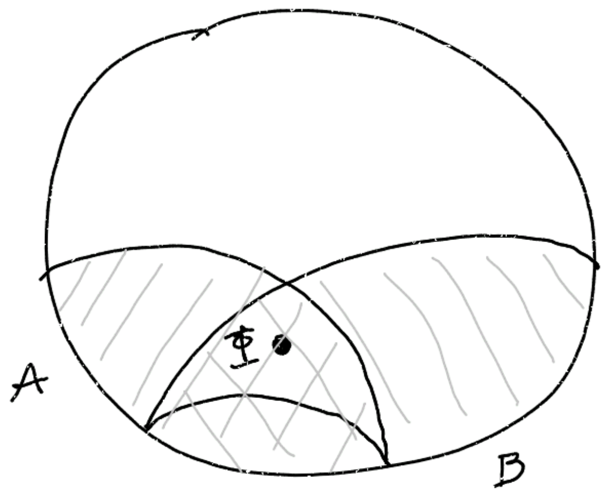
support localized to within Rindler wedge

Via AdS isometries, can localize support (i.e., transform Rindler wedge) to within causal wedge:



But now we arrive at an important puzzle: a given bulk field Φ may have different bndy representations; that is, the corresponding bndy op. is <u>not</u> unique!

Consider following 2 examples from 1411.7041 (ADH):



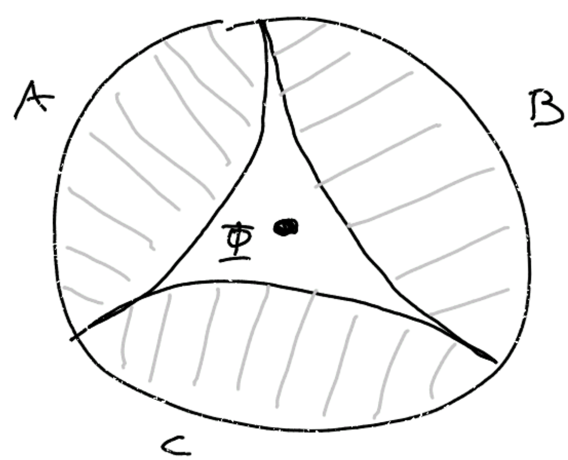
Via HKLL, Φ given by (nonlocal) bndy operator ϕ_A in A and ϕ_B in B.

For $\phi_A = \phi_B$, bndy support of dual op. must be localized in $A \cap B \longrightarrow \phi_{A \cap B}$.

But Φ lies outside causal wedge of $A \cap B$! $\therefore \phi_A \neq \phi_B$.

<u>2nd example:</u>

Φ cannot be reconstructed in A, B, or C.



Rather, have 3 bndy duals

$\varphi_{A\cup B}$, $\varphi_{B\cup C}$, $\varphi_{A\cup C}$, which cannot

possibly be equal (since $A \cap B \cap C$ 3 points; rotate + repeat to get mutual intersect. ϕ).

ADH conclusion: bulk-bndy mapping a form of QEC (i.e., AdS/CFT functions like a quantum secret sharing scheme).

E.g., 3 qutrit code:

Alice wishes to send $|\psi\rangle = \sum_{i=0}^{2} a_i |i\rangle$.

Instead, send $|\tilde{\psi}\rangle = \sum_{i=0}^{2} a_i |\tilde{i}\rangle$, where

$$\begin{cases} |\tilde{0}\rangle = \frac{1}{\sqrt{3}}\left(|000\rangle + |111\rangle + |222\rangle\right) \\ |\tilde{1}\rangle = \frac{1}{\sqrt{3}}\left(|012\rangle + |120\rangle + |201\rangle\right) \\ |\tilde{2}\rangle = \frac{1}{\sqrt{3}}\left(|021\rangle + |102\rangle + |210\rangle\right) \end{cases}$$

"logical" or "code subspace"

physical $\mathcal{H}$ (qutrits)

Neat properties:

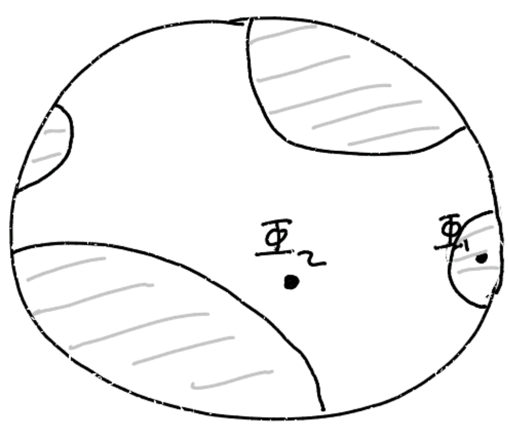
(1) Reduced density matrix on any qutrit max. mixed $\Rightarrow$ intercepting 1 qutrit gives no info about message.

(2) QEC: even if 1 qutrit is lost (error), Bob can still reconstruct state from remaining 2 qutrits.

$\rightarrow$ see ADH 1411.7041 or Fernando's tutorial for more details.

Applied to holography: local bulk operators as logical operations on encoded subspace, which are increasingly well-protected deeper in bulk:

see next page...



Φ_1: easily erased

Φ_2: very robust against large "errors" (missing bndy info).

Code subspace $\mathcal{H}_c$ = low-energy subspace of full $\mathcal{H}$. Why? Each bulk op Φ raises energy of state; eventually back-reaction important + perturbation theory invalid.

→ In language of QEC: Larger code subspace ⇒ fewer correctable errors.

Resolution of above puzzle: ϕ_{AB}, ϕ_{BC}, ϕ_{AC} are different operators in the physical (boundary) $\mathcal{H}$, but yield the same logical/code state Φ when acting on the bulk $\mathcal{H}_c$.

This idea has been elucidated in toy models involving tensor networks, notably in 1503.06237 by Fernando et al.

(more on tensor networks in Román Orús' GQFI lecture).

Why should AdS/CFT operate as a QEC?

Open question, but gauge invariance may play a fundamental role (1602.04811).

see also v. recent review: 1711.07787

Drawbacks of HKLL:

(1) Assumes bulk geo: need to solve bulk e.o.m. $(\Box - m^2)\Phi = 0$

contrary to philosophy of reconstruction, or emergent spacetime in general

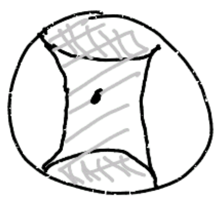
(2) Requires $1/N$ corrections, which are (very) messy + hard; can only reconstruct perturbatively, but ideally want non-perturbative def. of bulk Φ.

$\longrightarrow$ Related: finite $N \Rightarrow$ interactions

(3) Gauge-invariant operators must be suitably dressed

(4) Difficulties for non-trivial spacetimes: can only reconstruct within causal wedge → holographic shadows (a general obstacle: 1412.5175)

[(5) Reasons to believe entanglement wedge is more appropriate object]

(E.g.  Φ contributes to entanglement entropy of region, so should be captured by bndy dual.)

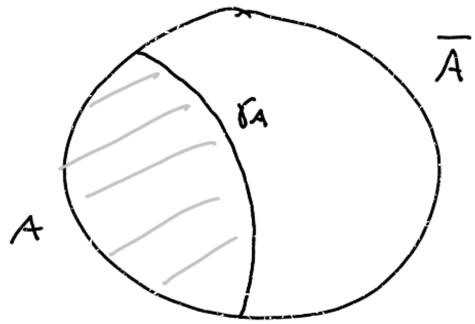
Ryu-Takayanagi (RT)

Previous lecture: AdS-Schwar. $\longleftrightarrow$ $|TFD\rangle$ suggests connection between (entanglement) entropy + (spacetime) geometry.

Precise generalization by RT in hep-th/0603001: (1622 citations)

$$S_A = \frac{\text{Area}(\gamma_A)}{4G_N}$$

time-slice of global AdS_3 :



$\longrightarrow$ entropy of subsystem in CFT corresponds to (min.) surface in AdS.

$\therefore$ complete knowledge of entanglement entropy* in CFT allows one to reconstruct bulk geometry!!

$\left\{ \begin{array}{l} \text{"spacetime from entanglement"} \\ \text{or "It from Qubit"} \end{array} \right\}$

(* $S(\rho_A) = -\mathrm{tr}\, \rho_A \ln \rho_A$, $\rho_A = \mathrm{tr}_{\bar{A}}\, \rho$

for all subregions A)

Quick review of entanglement:

- pure state $|\psi\rangle \in \mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$

$$\rho = |\psi\rangle\langle\psi| \longrightarrow S(\rho) = 0$$

e.g. $$\rho = \frac{1}{\sqrt{2}}\left(|1\rangle_A \otimes |1\rangle_B + |0\rangle_A |0\rangle_B\right)$$

$$= \frac{1}{\sqrt{2}}\left(|11\rangle + |00\rangle\right)$$

ρ <u>entangled</u> state:

consider subsystem A: $\rho_A = tr_B\, \rho$

$$= \sum_{\tilde{i}} {}_B\langle\tilde{i}|\psi\rangle\langle\psi|\tilde{i}\rangle_B$$

$$= {}_B\langle\tilde{0}|\psi\rangle\langle\psi|\tilde{0}\rangle_B + {}_B\langle\tilde{1}|\psi\rangle\langle\psi|\tilde{1}\rangle_B$$

$$= \frac{1}{2}\, {}_B\langle\tilde{0}|\Big(|11\rangle\langle 11| + |00\rangle\langle 00|$$
$$+ |11\rangle\langle 00| + |00\rangle\langle 11|\Big)|\tilde{0}\rangle_B$$

$$+ \frac{1}{2}\, {}_B\langle\tilde{1}|\left(\cdots\right)|\tilde{1}\rangle_B$$

$$= \frac{1}{2}\left(|0\rangle_A\, {}_A\langle 0| + |1\rangle_A\, {}_A\langle 1|\right) = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$S(\rho_A) = -\operatorname{tr} \rho_A \ln \rho_A$$

$$= -2 \cdot \tfrac{1}{2} \ln \tfrac{1}{2} = \ln 2$$

$\to$ Entanglement entropy (EE) gives # of entangled bits between subsystems

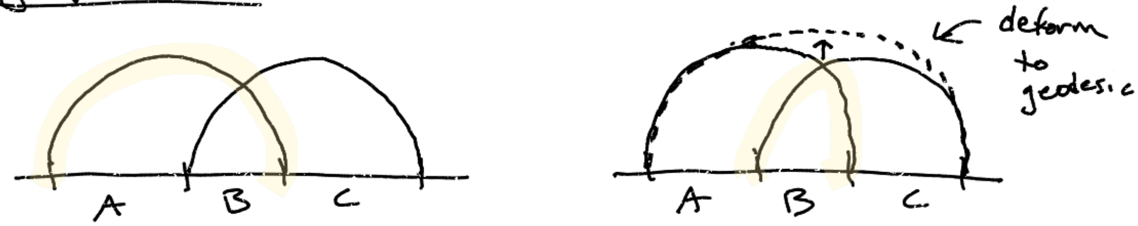
General: k entangled qubits $\Rightarrow S = k \ln 2$

Entropy obeys certain properties; important e.g., SSA

$$S_{ABC} + S_B \leq S_{AB} + S_{BC}$$

$$S_{AB} \equiv S(\rho_{A \cup B})$$

Holographic proof of SSA (via RT):

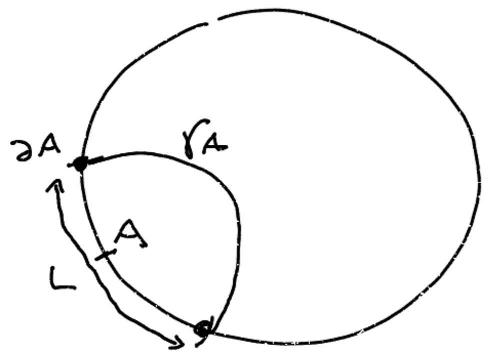


$$\text{Area}(A \cup B) + \text{Area}(B \cup C) \geq \text{Area}(B) + \text{Area}(A \cup B \cup C)$$

RT $\Rightarrow$ Area $\sim$ entropy, QED

Concrete example: AdS_3 (in Poincaré)

$$ds^2 = \frac{\ell^2}{z^2}\left(dz^2 - dt^2 + dx^2\right)$$



- Consider region A of radius $R = L/2$
- Find min. bulk surface γ_A homologous to A

E.o.m $\Rightarrow$ geodesic half-circle

$$(z, x) = \frac{L}{2}(\sin\lambda, \cos\lambda), \quad \lambda \in [a, \pi - a]$$

where $a = \frac{2\varepsilon}{L}$, ε IR cutoff

$$\left(\text{acc} \Rightarrow \begin{cases} z = \frac{L}{2}\sin a \sim \frac{La}{2} = \varepsilon & (\min z) \\ x = \frac{L}{2}\cos a \sim \frac{L}{2} & (\text{radius of ball}) \end{cases} \right)$$

$$\text{Area}(\gamma_A) = \text{length}(\gamma_A) = \frac{2\ell}{L}\ln\left(\frac{L}{\varepsilon}\right)$$

$$\text{RT}: \quad S(A) = \frac{\text{Area}(\gamma_A)}{4G_N} = \underline{\frac{\ell}{2G_N}\ln\left(\frac{L}{\varepsilon}\right)}$$

For CFT_2, QFT calculation

via "replica trick" yields $S_L = \frac{c}{3} \ln\left(\frac{L}{\varepsilon}\right)$

UV cutoff

$\Longrightarrow$ bulk/bndy match if identify

$$\boxed{c = \frac{3\ell}{2G_N}}$$

known from independent computations

(e.g. thermal entropy) $\Rightarrow$ check of RT ✓

Note also dual role of ε: UV $\longleftrightarrow$ IR!

General proof by LM: 1304.4926

(pedagogical: sec. 2 of 1310.5713)

Reviews of holographic EE:

1609.00026, 0905.0932

Open questions:

- How is spacetime geo. encoded in EE?
  (how to get "It" from "qubit")

  $\longrightarrow$ deeper relation between entanglement and geometry (e.g. Einstein eqns 1308.3716)

- what is role / reason for QEC?

- How to reconstruct in non-trivial spacetimes (e.g. holographic shadows, BH interiors)?

- Elucidate rich + fascinating connections between $\underbrace{\text{gravity, QFT, and information}}_{\text{GQFI!}}$